On ω-power languages

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The infinite or ω-power is one of the basic operations to associate with a language of finite words (a finitary language) an ω-language.

It plays a crucial role in the characterization of regular and of context-free ω-languages, that is, ω-languages accepted by (nondeterministic) finite or pushdown automata, respectively (cf. the surveys [St87a, Th90]). But in connection with the determinization of finite ω-automata it turned out that the properties of the ω-power are remarkable elusive; resulting in the well-known complicated proof of MACNAUGHTON's theorem [MN66]. Later work [TB70, Ei74, Ch74] showed a connection between the ω-power of regular ω-languages and a limit operation (called here δ-limit) transferring languages to ω-languages. It was, therefore, asked in [Ch74] for more transparent relationships between the ω-power and the δ-limit of languages. It turned out that this δ-limit is a useful tool in translating the finite to the infinite behaviour of deterministic accepting devices (cf. [Li76, CG78, St87a, Th90, EH93]).

As it was mentioned above ω-power languages play a crucial role in the characterization of ω-languages accepted by nondeterministic finite or push-down automata. In fact, they are useful in general for the characterization of ω-languages accepted by empty storage (cf. [St77]).

Therefore, a general relationship between ω-power and δ-limit could hint for instances where ω-languages accepted nondeterministically via empty-storage-acceptance could be likewise accepted deterministically.

In contrast to the ω-power the δ-limit yields, similar to the adherence of languages, a transparent description of the ω-language derived from the language. Particularly remarkable are the facts that in terms of the natural CANTOR-topology of the space of ω-words it describes exactly $\mathbf{G}_\delta$-sets (This being also the reason for calling it δ-limit.) and, moreover, it allows for a specification of the topological (BOREL-) subclasses of $\mathbf{G}_\delta$ in terms of the underlying (preimage-)languages (cf. [St87b]).

No such properties, however, are known in general for ω-power languages. Except for the representation as an infinite product, the ω-power of a language W, W^ω, is known to be the maximum solution of a linear homogenuous equation in one variable (see Eq. (H) below). The disadvantage of those equations is, in contrast to the language case, that they are not uniquely solvable in ω-languages. This is, however, no obstacle to obtain an axiom system for ω-regular expressions similar to the one for regular expressions given by A. SALOMAA in [Sa66]. K. WAGNER [Wa76] showed that the maximum solution principle of [Re72, St72] is sufficient for this purpose.[1]

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[1] Other axiom systems for ω-regular expressions were given in [DK84, II84]

Therefore, we start our investigations with the consideration of linear equations for ω-languages. After introducing some necessary notation in the first section we derive the above mentioned maximum solution principle and some conditions under which equations are equivalent, that is, have the same set of solutions.

In the second section we consider the structure of the set of solutions of a linear equation. To this end we introduce the notion of atomic solutions, that is, nonempty solutions which are in some sense indivisible. Section 3 is devoted to the case when a linear equation has a finite-state or even regular solution. This concludes our investigations on solutions of linear equations, and we turn to the consideration of ω-power languages.

In the fourt part we deal with relationships between the operations of ω-power and δ-limit. Thereby it is natural to consider also toplogical properties of ω-power languages. It turns out that already for topological reasons the δ-limit is not able to describe all ω-power languages.
Moreover, we show which ω-power languages can be found in several low level BOREL-classes (below the class $\mathbf{G}_\delta$). Here the behaviour of ω-power languages is in contrast to the class of so-called strongly-connected ω-languages (cf. [St80a, 83]). Strongly-connected ω-languages are already closed if they are in the BOREL-class $\mathbf{F}_\sigma \cap \mathbf{G}_\delta$, whereas we derive as well examples of open nonclosed as examples of nonopen and nonclosed ω-power languages in $\mathbf{F}_\sigma \cap \mathbf{G}_\delta$.

The final section of this paper deals with another topological property of ω-power languages. It was observed in [St76, 80b] that finite-state (or regular) ω-languages which are nowhere dense in CANTOR-space lack some subword (finite pattern). Here we generalize this result to finite-state ω-languages nowhere dense in an ω-power language.

1 Linear Equations

In this section we introduce some notation used throughout the paper. Further we give some basic results from the theory of ω-languages which are necessary for our investigations. Additional information on the theory of ω-languages can be obtained from the quoted above papers.

After these preparations we introduce linear equations for ω-languages and show how to solve them. An especially interesting way of solving is the maximum solution principle which will be illustrated at the end of this section by two examples. Moreover, we derive a condition under which equations have the same set of solutions.

By $\mathbb{N} = \{0,1,2,\ldots\}$ we denote the set of natural numbers. We consider the space X^ω of infinite strings (sequences) on a finite alphabet of cardinality $card X \geq 2$. By X^* we denote the set (monoid) of finite strings (words) on X, including the *empty* word e. For $w \in X^*$ and $b \in X^* \cup X^\omega$ let $w \cdot b$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. As usual we denote subsets of X^* as *languages* and subsets of X^ω as ω-*languages*. For a language $W \subseteq X^*$ let $W^0 := \{e\}$ and $W^{i+1} := W^i \cdot W$. Then $W^* := \bigcup_{i \in \mathbb{N}} W^i$ is the *submonoid* of X^* generated by W, and by W^ω we denote the set of infinite strings formed by concatenating in W. Furthermore $|w|$ is the *length* of the word $w \in X^*$.

$\mathbf{A}(B) := \{w : w \in X^* \wedge \exists b(b \in X^* \cup X^\omega \wedge w \cdot b \in B)\}$ is the set of all *initial words (prefixes)* of the set $B \subseteq X^* \cup X^\omega$. For the sake of brevity we shall write $w \cdot B, W \cdot b$ and $\mathbf{A}(b)$ instead of $\{w\} \cdot B, W \cdot \{b\}$ and $\mathbf{A}(\{b\})$ respectively, and we shall abbreviate the fact that w is an initial word of b, that is $w \in \mathbf{A}(b)$, by $w \sqsubseteq b$. Moreover, we call $B \subseteq X^* \cup X^\omega$ *prefix-free* iff $w \sqsubseteq b$ and $w, b \in B$ imply $w = b$, a prefix-free subset $C \subseteq X^* \setminus \{e\}$ is also called a *prefix code*.

We consider X^ω as a topological space with the basis $(w \cdot X^\omega)_{w \in X^*}$. Since X is finite, this topological space is homeomorphic to the CANTOR discontinuum, hence compact. In the asequel we shall refer to the space X^ω also as CANTOR-*space*. Open sets in X^ω are of the form $W \cdot X^\omega$ where $W \subseteq X^*$. From this follows that a subset $F \in X^\omega$ is *closed* iff $\mathbf{A}(\beta) \subseteq \mathbf{A}(F)$ implies $\beta \in F$.

The *topological closure* of subset $F \subseteq X^\omega$, that is, the smallest closed subset of (X^ω, ρ) containing F is denoted by $\mathcal{C}(F)$. It holds $\mathcal{C}(F) = \{\xi : \mathbf{A}(\xi) \subseteq \mathbf{A}(F)\}$.

Having defined open and closed sets in X^ω, we proceed to the next classes of the Borel hierarchy (cf. [Ku66]):

$\mathbf{F}_\sigma$ is the set of countable unions of closed subsets of X^ω, and

$\mathbf{G}_\delta$ is the set of countable intersections of open subsets of X^ω.

For $W \subseteq X^* \setminus \{e\}$ and $E \subseteq X^\omega$ we consider the equations

$$T = W \cdot T \tag{H}$$

and

$$T = W \cdot T \cup E \tag{I}$$

which will be referred to as the *homogenuous* and *inhomogenuous* equations, respectively.

It was already observed by TRAKHTENBROT [Tr62] that the simple equation $T = X \cdot T$ has uncountably many ω-languages as solutions (cf. also [St83]). Therefore, in this section and the subsequent ones we address the problem which subsets of X^ω are solutions of the given equations.

From [St72] and [Re72] the following simple properties are known. Let $W \subseteq X^* \setminus \{e\}$ and $E \subseteq X^\omega$. Then

$$\begin{aligned}
F \subseteq W \cdot W^* \cdot F \quad &\text{implies} \quad F \subseteq W^\omega, && (1)\\
W \cdot F \cup E \subseteq F \quad &\text{implies} \quad W^* \cdot E \subseteq W^* \cdot F \subseteq F, && (2)\\
W \cdot F \cup E = F \quad &\text{implies} \quad W^* \cdot E \subseteq F \subseteq W^\omega \cup W^* \cdot E, \text{ and} && (3)\\
\text{If } F = W \cdot F \quad &\text{then} \quad F \cup W^* \cdot E \text{ is a solution of Eq. (I).} && (4)
\end{aligned}$$

Moreover, it was observed that $W^* \cdot E$ as well as $W^\omega \cup W^* \cdot E$ are solutions of Eq. (I), according to Eq. (3) they are the *minimum* and *maximum* solution, respectively. As a corollary to the above properties we get the maximum solution principle which has been proved useful in establishing identities involving ω-power languages W^ω (e.g. in [Lt88, 91a, 91b, St80a, Wa76].

Corollary 1 (Maximum solution principle)
Let $F \subseteq X^\omega$ satisfy Eq. (I), and let $W^\omega \subseteq F$. Then

$$F = W^\omega \cup W^* \cdot E \ .$$

As a further corollary to Eq. (1) we get

Corollary 2 *If $F \subseteq W \cdot F$ then $W^* \cdot F$ is the minimum solution of the homogenuous equation Eq. (H) containing F.*

This yields the following relation between the solutions of the inhomogenuous and homogenuous converse to Eq. (4):

Lemma 3 *If $F = W \cdot F \cup E$ then $F' := W^* \cdot (F \setminus W^* \cdot E)$ is the minimum solution of the homogenuous equation Eq. (H) such that $F = F' \cup W^* \cdot E$.*

Proof. We have $F \setminus W^* \cdot E = (W \cdot F \cup E) \setminus W^* \cdot E = W \cdot F \setminus W^* \cdot E \subseteq W \cdot (F \setminus W^* \cdot E)$, and the assertion is immediate with Corollary 2. ❑

Next we consider pairs of coefficients (W,E) and (V,E') to be *equivalent* if the inhomogenuous equations $T = W \cdot T \cup E$ and $T = V \cdot T \cup E'$ have the same set of solutions. We obtain the following.

Lemma 4 *Let $W \cup V \subseteq X^* \setminus \{e\}$, $W^* = V^*$, and $W^* \cdot E = V^* \cdot E'$. Then (W,E) and (V,E') are equivalent.*

Proof. If F is a solution of Eq. (I) then $W^* \cdot F = F$. This together with the inclusion $W^* \cdot E \subseteq F$ and the identity $W^* = (W \cdot W^*)^*$ yields $F = W \cdot W^* \cdot F \cup W^* \cdot E$.

Conversely, if $F = W \cdot W^* \cdot F \cup W^* \cdot E = W \cdot (W^* \cdot F \cup W^* \cdot E) \cup E$ in virtue of $W^* = (W \cdot W^*)^*$ we have $F = W^* \cdot F$ and $W^* \cdot E \subseteq F$. Thus $F = W \cdot (W^* \cdot F \cup W^* \cdot E) \cup E = W \cdot F \cup E$, and (W,E) is equivalent to $(W \cdot W^*, W^* \cdot E)$. In the same way (V,E') is equivalent to $(V \cdot V^*, V^* \cdot E')$, and the assertion follows. ❑

Corollary 5 *Let $e \notin W$ and $W^n \subseteq V \subseteq W \cdot W^*$. Then $F = W \cdot F \cup E$ implies $F = V \cdot F \cup W^* \cdot E$.*

Proof. From $F = W \cdot F \cup E$ we have the identity $F = W \cdot F \cup W^* \cdot E$. Inserting the right hand side of this identity n times into itself yields $F = W^n \cdot F \cup W^* \cdot E$. On the other hand $F = W \cdot W^* \cdot F \cup W^* \cdot E$, and the assertion follows. ❑

The converse statement, however, is not valid. Consider e.g. $W := \{a,b\}, V := \{a,b\}^2$, $F := V^* \cdot \{aa,ba\}^\omega$. Then $F = V \cdot F$ but $F \neq \{a,b\} \cdot F$ because $(ab)^\omega \notin F$.

As it was announced above we conclude this section with two instances whose proofs show the usefulness of the simple properties derived in Eqs. (1) … (4) and Corollary 1 when solving equations like Eq. (H) or Eq. (I).

To every instance we need some preparatory definitions.

As in [Lt88] or [St80] we define the *stabilizer* of an ω-language $E \subseteq X^\omega$,

$$Stab(E) := \{w : w \in \mathbf{A}(E) \setminus \{e\} \wedge w \cdot E \subseteq E\} . \tag{5}$$

Since $\mathcal{C}(w \cdot E) = w \cdot \mathcal{C}(E)$, we have $Stab(E) \subseteq Stab(\mathcal{C}(E))$. Moreover, the stabilizer of an ω-language $E \subseteq X^\omega$, $Stab(E)$, is closed under concatenation, that is, is a subsemigroup of X^*.

Obviously, the stabilizer of an ω-power language W^ω satisfies $W^* \setminus \{e\} \subseteq Stab(W^\omega) \subseteq Stab(\mathcal{C}(W^\omega)) \subseteq \mathbf{A}(W^\omega)$ and $Stab(W^\omega) \cdot W^\omega = W^\omega$.

We obtain a result which is similar to the construction of a minimal generator of the semigroup W^*, $(W \setminus \{e\}) \setminus ((W \setminus \{e\}) \cdot (W^* \setminus \{e\}))$.

Theorem 6 ([Lt88, Proposition IV.3][2]) *Let $W \subseteq X^* \setminus \{e\}$ and let $V := W \setminus (W \cdot Stab(W^\omega))$. Then $V^\omega = W^\omega$.*

Proof. The inclusion $V^\omega \subseteq W^\omega$ follows from $V \subseteq W$.

On the other hand, we have $V \cdot Stab(W^\omega) \supseteq W^* \setminus \{e\}$. Thus $V \cdot W^\omega = V \cdot Stab(W^\omega) \cdot W^\omega \supseteq W^* \cdot W^\omega = W^\omega$, and Eq. (1) implies $W^\omega \subseteq V^\omega$. ❏

It should be noted that, in contrast to the minimal generator of W^* the language $V := W \setminus (W \cdot Stab(W^\omega))$ defined in Theorem 6 need not be a minimal ω-generator of W^ω contained in $W \subseteq X^* \setminus \{e\}$. In [Lt88, Example IV.4] and [Lt91a, Example 1] it is shown that $W = a \cdot b^* \cup ba \cdot b^*$ satisfies $W = W \setminus (W \cdot Stab(W^\omega))$, but $W^\omega = (W \setminus \{ab\})^\omega$.

Next we derive an instance where the maximum solution principle is used. We give an explicit formula for the closure of W^ω, $\mathcal{C}(W^\omega)$.

To this end let $ls\,W := \{\xi : \xi \in X^\omega \wedge \mathbf{A}(\xi) \subseteq \mathbf{A}(W)\}$ be the *adherence* of the language $W \subseteq X^*$. Then it is known that $\mathcal{C}(W \cdot E) = W \cdot \mathcal{C}(E) \cup ls\,W$ when $W \subseteq X^*$, $E \subseteq X^\omega$ and $E \neq \emptyset$. We obtain the formula

$$\mathcal{C}(W^\omega) = W^\omega \cup W^* \cdot ls\,W . \tag{6}$$

Proof. Since $\mathcal{C}(W^\omega)$ is the closure of W^ω, we have $\mathcal{C}(W^\omega) \supseteq W^\omega$. Now $W^\omega = W \cdot W^\omega$, and from the formula mentioned above we get $\mathcal{C}(W^\omega) = \mathcal{C}(W \cdot W^\omega) = W \cdot \mathcal{C}(W^\omega) \cup ls\,W$. Our assertion follows from Corollary 1. ❏

2 Atomic solutions of the homogenuous equation

In view of Eq. (4) and Lemma 3 every solution of the inhomogenuous equation can be obtained by adding $W^* \cdot E$ to a solution of the homogenuous equation. In this section we, therefore, analyze the structure of the set of solutions of Eq. (H).

To this end we consider nonempty solutions of which are in some sense minimal. We call a nonempty solution S of the homogenuous equation *atomic* if it does not contain two nonempty disjoint solutions of Eq. (H). It is obvious that a nonempty

[2] cf. also [Lt91a, Lemma 2] or [Lt91b, Lemma 2.1].

and minimal (with respect to set inclusion) solution of Eq. (H) is atomic, but as we shall see below the converse is not true.

In order to construct atomic solutions we consider so-called W-factorizations of ω-words $\xi \in W^\omega$. A *W-factorization* is a factorization $\xi = w_0 \cdot w_1 \cdots w_i \cdots$ where $w_i \in W \setminus \{e\}$.

Theorem 7 *For every $\beta \in W^\omega$ there is an atomic solution of Eq. (H) containing β.*

Proof. Let $\beta = w_0 \cdot w_1 \cdots w_i \cdots$ be a W-factorization of β and define $\beta_j := w_j \cdot w_{j+1} \cdots w_i \cdots$, that is, $\beta_0 := \beta$ and $\beta_j = w_j \cdot \beta_{j+1}$.

It is easy to verify that $S := W^* \cdot \{\beta_j : j \in \mathbb{N}\}$ is a solution of Eq. (H). It remains to show that S is atomic. Assume $S_1, S_2 \subseteq S$, $S_1 \cap S_2 = \emptyset$ and $W \cdot S_m = S_m$ $(m = 1, 2)$.

If $\{\beta_j : j \in \mathbb{N}\} \subseteq S_1$ then $S_2 = \emptyset$. So let $\beta_{j_m} \in S_m$ and $j_2 < j_1$ (say). Since $W^* \cdot S_1 = S_1$, it follows $\beta_{j_2} \in S_1$, a contradiction to $S_1 \cap S_2 = \emptyset$. ❑

The proof of Theorem 7 provides us with a method for constructing atomic solutions of the homogenuous equation.

Corollary 8 *Let $\beta = w_0 \cdot w_1 \cdots w_i \cdots$ be a W-factorization of β. Then for every infinite subset $M \subseteq \mathbb{N}$ the set $S_M := W^* \cdot \{\beta_j : j \in M\}$ is a solution of Eq. (H).*

From the above described construction of atomic solutions the following description of arbitrary solutions is obvious.

Lemma 9 *If $F = W \cdot F$ then F is the union of all atomic solutions of Eq. (H) contained in F.*

Though it is not easy, in general, to obtain a concise description of atomic solutions containing $\beta \in W^\omega$, for ultimately periodic ω-words we have the following.

Property 10 *Let $\beta \in X^\omega$ be ultimately periodic. Then every atomic solution of Eq. (H) containing β has the form $W^* \cdot v^\omega$ for an appropriate $v \in W^* \setminus \{e\}$. Conversely, every ω-language $W^* \cdot v^\omega$ where $v \in W^* \setminus \{e\}$ is an atomic solution of Eq. (H).*

Proof. Let $\beta = w \cdot u^\omega$, and let $\beta = w_0 \cdot w_1 \cdots w_i \cdots$ be a W-factorization of β. Then there are infinitely many $j \in \mathbb{N}$ such that $\beta_j = \hat{v}^\omega$ for some $\hat{v} \neq e$. Following Corollary 8 the set $W^* \cdot \hat{v}^\omega$ is an atomic solution of Eq. (H).

We have still to show that $\hat{v}^\omega = v^\omega$ for some $v \in W^* \setminus \{e\}$. To this end observe that if $\beta_j = \beta_k = \hat{v}^\omega$ and $j < k$ then $\beta_j = v \cdot \beta_k$ for an appropriate $v \in W^* \setminus \{e\}$, whence $\beta_j = v^\omega$.

The second assertion is obvious. ❑

Atomic solutions containing a given β, however, may be neither minimal nor unique. Lemma 9 and the proof of Theorem 7 yield only the following sufficient conditions.

Property 11 *If S is a unique atomic solution containing an ω-word β then S is the unique minimal solution containing β.*

Property 12 *If $\beta \in W^\omega$ has a unique W-factorization $\beta = w_0 \cdot w_1 \cdot \ldots \cdot w_i \cdot \ldots$ $(w_i \in W)$ then the atomic solution of Eq. (H) containing β is unique.*

Remark: The latter condition is not necessary. Consider e.g. the suffix code $C := \{b, ba, aa\}$. Here ba^ω has two C-factorizations $ba^\omega = b \cdot aa \cdot aa \cdot \ldots = ba \cdot aa \cdot \ldots$ but $C^* \cdot a^\omega$ is the unique atomic solution of the equation $T = C \cdot T$ containing ba^ω.

The following example shows that atomic solutions containing a particular ω-word β may not be unique, even if W is a code[3] and β is ultimately periodic. In addition this example verifies that, though β has more than one W-factorizations all atomic solutions containing β are minimal.

Example 1 *Consider the suffix code $W_1 = \{ab, ba, baa\}$, and let $\beta_1 := baa(ba)^\omega = ba(ab)^\omega$. By Property 10, $W_1^* \cdot (ba)^\omega$ and $W_1^* \cdot (ab)^\omega$ are the only atomic solutions containing β_1. Obviously they are incomparable, thus minimal.*

Their intersection $W_1^* \cdot (ba)^\omega \cap W_1^* \cdot (ab)^\omega = W_1^* \cdot \beta_1$ does not contain a solution of Eq. (H), because neither $(ab)^\omega \in W_1^* \cdot \beta_1$ nor $(ba)^\omega \in W_1^* \cdot \beta_1$.

We add an example that atomic solutions need not be minimal.

Example 2 *Let $W_2 := \{aba, ba, baa\}$ (which is not a code). Then $\beta_2 := (baa)^\omega = ba \cdot (aba)^\omega$ yields the following two atomic solutions $W_2^* \cdot (baa)^\omega$ and $W_2^* \cdot (aba)^\omega$. One easily verifies that $(aba)^\omega \notin W_2^* \cdot (baa)^\omega$ whereas $(baa)^\omega \in W_2^* \cdot (aba)^\omega$. Hence $W_2^* \cdot (baa)^\omega \subset W_2^* \cdot (aba)^\omega$, and the latter atomic solution is not a minimal one.*

Atomic solutions are countable subsets of X^ω, hence, as countable unions of closed sets, $\mathbf{F}_\sigma$-sets. Thus Eq. (H) has (if ever) among its nonempty solutions always $\mathbf{F}_\sigma$-sets. Topologically simpler sets than $\mathbf{F}_\sigma$-sets are closed sets. But for Eq. (I) and Eq. (H) it turns out that they have at most one nonempty closed set as solution.

Lemma 13 *Let $W \neq \emptyset$. Then Eq. (I) has a nonempty closed solution iff $ls\,W \cup \mathcal{C}(E) \subseteq W^\omega \cup W^* \cdot E$, and moreover this solution is the maximum solution.*

Proof. First observe that similar to Eq. (6) the closure of the maximum solution $W^\omega \cup W^* \cdot E$ is calculated as $\mathcal{C}(W^\omega \cup W^* \cdot E) = W^\omega \cup W^* \cdot ls\,W \cup W^* \cdot \mathcal{C}(E)$, and it satisfies $\mathcal{C}(W^\omega \cup W^* \cdot E) \subseteq W^\omega \cup W^* \cdot E$ if $ls\,W \cup \mathcal{C}(E) \subseteq W^\omega \cup W^* \cdot E$.

On the other hand if $W^\omega \cup W^* \cdot E = \mathcal{C}(W^\omega \cup W^* \cdot E) = W^\omega \cup W^* \cdot ls\,W \cup W^* \cdot \mathcal{C}(E)$ the condition is trivially satisfied.

The second assertion is obvious from $W^\omega \subseteq \mathcal{C}(W^* \cdot F)$ whenever $W^* \cdot F$ is nonempty. ❏

As a corollary we obtain a necessary and sufficient condition for an ω-power language W^ω to be closed.

Corollary 14 *An ω-power language $W^\omega \subseteq X^\omega$ is closed if and only if $ls\,W \subseteq W^\omega$.*

We conclude this section with a lower estimate for the possible number of solutions of the inhomgenuous equation Eq. (I). To this end we derive an intersection property.

[3]That is, for all words $v_1, \ldots, v_\ell, w_1, \ldots, w_m \in W$ the identity $v_1 \cdots v_\ell = w_1 \cdots w_m$ implies $\ell = m$ and $v_i = w_i$ $(i = 1, \ldots, \ell)$.

Lemma 15 *Let $F = V \cdot F$ and $E = W \cdot E$ where $V \subseteq W^* \setminus \{e\}$. If every ω-word $\xi \in E \cap F$ has at most one W-factorization then $F \cap E = V \cdot (F \cap E)$.*

Proof. Since $V \subseteq W^*$, the inclusion $V \cdot (F \cap E) \subseteq F \cap E$ is immediate. To prove the converse we use that every $\beta \in F \cap E$ has a unique W-factorization. First $F = V \cdot F$ implies that $\beta = v \cdot \xi$ for some $v \in V \subseteq W^*$ and $\xi \in F \subseteq W^\omega$.

Let $v = v_1 \cdots v_n$ and $\xi = w_0 \cdot w_1 \cdots w_i \cdots$ where $v_j, w_i \in W \setminus \{e\}$. Thus $\beta = v_1 \cdots v_n \cdot w_0 \cdot w_1 \cdots w_i \cdots$ is the unique W-factorization of β. As $v \in W^n$ and $E = W^n \cdot E$, it follows that $\xi \in E$. Hence $\beta = v \cdot \xi \in V \cdot (E \cap F)$. ❏

As a second preparation we derive TRAKHTENBROT'S [Tr62] description of all atomic solutions of the equation $T = X \cdot T$.

Example 3 (Atomic solutions of $T = X \cdot T$) *Utilizing the technique of the proof of Theorem 7 we observe that for the equation $T = X \cdot T$ and $\beta \in X^\omega$ it holds $\{\beta_i : i \in \mathbb{N}\} = \mathbf{E}(\beta)$ where $\mathbf{E}(\beta)$ is the set of all tails of β. Hence $F_\beta := X^* \cdot \mathbf{E}(\beta)$ is the (unique, according to Property 12) atomic solution of $T = X \cdot T$ containing β.*

Consequently, either $F_\beta = F_\xi$ or $F_\beta \cap F_\xi = \emptyset$.

Theorem 16 *If the cardinality of the set $W^\omega \setminus W^* \cdot E$ satisfies $\operatorname{card} W^\omega \setminus W^* \cdot E = 2^{\aleph_0}$ then Eq. (I) has $2^{2^{\aleph_0}}$ solutions.*

Proof. Clearly, Eq. (I) can have no more than $2^{2^{\aleph_0}}$ solutions.

According to Lemma 3 the set $F' := W^* \cdot (W^\omega \setminus W^* \cdot E)$ is the minimum solution of Eq. (H) such that $W^\omega = F' \cup W^* \cdot E$. Now applying Lemma 15 and the fact that each one of the sets F_β defined in Example 3 is countable we obtain that the set $\{F' \cap F_\beta : \beta \in W^\omega \setminus W^* \cdot E\}$ is an uncountable family of pairwise disjoint solutions of Eq. (H). Hence, Eq. (H) has all unions $\bigcup_{\beta \in M}(F' \cap F_\beta)$ where $M \subseteq W^\omega \setminus W^* \cdot E$ as solutions, that is, it has at least $2^{2^{\aleph_0}}$ solutions. These solutions differ already on $W^\omega \setminus W^* \cdot E$ which proves that the family of all unions $\bigcup_{\beta \in M}(F' \cap F_\beta) \cup W^* \cdot E$ provides $2^{2^{\aleph_0}}$ solutions of Eq. (I). ❏

3 Regular and finite-state solutions

In this section we consider solutions of our equation which are closely related to the well-known class of regular ω-languages. To this end we introduce the following.

For a set $B \subseteq X^* \cup X^\omega$ we define the *state* B/w of B generated by the word $w \in X^*$ as $B/w := \{b : w \cdot b \in B\}$, and we call a set B *finite-state* if the number of different states B/w ($w \in X^*$) is finite. Finite-state languages $W \subseteq X^*$ are also known as *regular* languages. Already TRACHTENBROT [Tr62] (cf. also [St83]) observed that the class of finite-state ω-languages is much larger than the class of ω-languages accepted by finite automata (so-called regular ω-languages). An ω-language $F \subseteq X^\omega$ is referred to as *regular* provided there are regular languages $W_i, V_i \subseteq X^*$ ($i = 1, \ldots, n$) such that $F = \bigcup_{i=1}^n W_i \cdot V_i^\omega$.

LITOVSKY and TIMMERMAN [LT87] have shown that a regular ω-power language W^ω is already generated by a regular language L. In this section we consider the related case when the coefficients W and E of the inhomogenuous equation Eq. (I) are finite-state or even regular. Our general result follows.

Theorem 17 *If W and E are finite-state then every solution of Eq.* (I) *is also finite-state.*

Proof. First we mention that $W^* \cdot E$ is also finite-state if W and E are finite-state.
Let $W^{(n)} := \{w : w \in W^* \wedge |w| \geq n\}$. In view of Corollary 5 $F = W \cdot F \cup E$ implies $F = W^{(n)} \cdot F \cup W^* \cdot E$ for arbitrary $n \in \mathbb{N}$.
Next, we use the property that $W^{(|w|)}/w = W^*/w$. Then $F/w = (W^{(|w|)} \cdot F)/w \cup (W^* \cdot E)/w = (W^*/w) \cdot F \cup (W^* \cdot E)/w$. Thus the number of states of F is not larger than the product of the number of states of W^* and $W^* \cdot E$. ❑

In the rest of this section we verify that Theorem 17 does not hold in the case of regular sets, and that in order to have only finite-state solutions it is not sufficient to have one finite-state solution.
The first fact is easily verified by the equation $F = X \cdot F$ which has $2^{2^{\aleph_0}}$ solutions. Consequently, most of them cannot be regular.
Next we give an equation which has a regular minimum solution but its maximum solution is not finite-state.

Example 4 *Let $W_4 := \{a^{n!} \cdot b : n \in \mathbb{N}\}$ and $E_4 := \{a, b\}^* \cdot a^\omega$. Then Eq.* (I) *has the minimum solution $W_4^* \cdot E_4 = E_4$ which is regular, but its maximum solution $W_4^\omega \cup W_4^* \cdot E_4$ is not finite-state, because $(W_4^\omega \cup W_4^* \cdot E_4) \cap (a^* \cdot b)^\omega = W_4^\omega$ is not finite-state.*

Before proceeding to the next example we need the following lemma.

Lemma 18 *An ω-language of the form $V \cdot \beta$ is finite-state iff there are a regular language V' and a word u such that $V \cdot \beta = V' \cdot u^\omega$.*

Proof. Clearly, the condition is sufficient. Conversely, if $V \cdot \beta$ is finite-state then there are words w and w' such that $w' \neq e$, $w \cdot w' \sqsubset \beta$ and $(V \cdot \beta)/w \cdot w' \subseteq (V \cdot \beta)/w$. Let $\xi := \beta/(w \cdot w')$. Since $\xi \in V \cdot \beta/w$, we have $w \cdot \xi = v \cdot \beta$ for some $v \in V$. On the other hand, $w \cdot w' \cdot \xi = \beta$. Consequently, $w \cdot \xi = v \cdot w \cdot w' \cdot \xi$, that is, $\xi = u^\omega$ where $w \cdot u = v \cdot w \cdot w'$. Then β is also ultimately periodic. Now define $V' := \{w : \xi \in (V \cdot \beta)/w\}$. ❑

Example 5 *The equation $T = X \cdot T \cup \{\xi\}$ has always the finite-state (even regular) maximum solution $F_5 = X^\omega$, but according to our lemma its minimum solution $X^* \cdot \xi$ is finite-state if and only if ξ is ultimately periodic.*

4 ω-power and δ-limit

In the preceding sections we have seen that, in contrast to the case of languages, the linear equations Eq. (H) and Eq. (I) may have many solutions in the range of

ω-languages. One of the solutions of Eq. (H) and a particularly interesting one (its maximum solution) is the ω-power W^ω.

In this section we investigate properties of the ω-power operation and its relation to a limit operation mapping also languages to ω-languages—the so-called δ-*limit*[4] of a language $W \subseteq X^*$,

$$W^\delta := \{\zeta : \zeta \in X^\omega \wedge \mathbf{A}(\zeta) \cap W \text{ is infinite}\} . \tag{7}$$

We also consider topological properties of ω-power languages. In this section we investigate their relationships to BOREL-classes, and in the subsequent one we focus on (relative) density.

In connection with acceptance results for ω-languages, like those ones as MACNAUGHTON's theorem, properties of ω-power languages are remarkably elusive. In this respect the δ-limit has more transparent properties. Therefore it would be desirable to derive some relationships between the operations of ω-power and δ-limit. To this end we first calculate the δ-limit of the concatenation product and the KLEENE-star of languages (cf. [St80a],[Lt88]).

$$W \cdot V^\delta \subseteq (W \cdot V)^\delta \subseteq W \cdot V^\delta \cup W^\delta \tag{8}$$

Particular cases of Eq. (8) are obtained for $W^\delta = \emptyset$ or $e \in V$, respectively.

$$\begin{aligned} (W \cdot V)^\delta &= W \cdot V^\delta && \text{if} \quad W^\delta = \emptyset \text{ , and} && (9)\\ (W \cdot V)^\delta &= W \cdot V^\delta \cup W^\delta && \text{if} \quad e \in V && (10) \end{aligned}$$

In virtue of the obvious inclusion $W^\omega \subseteq (W^*)^\delta$ we obtain via the maximum solution principle Corollary 1 the following.

$$(W^*)^\delta = W^\omega \cup W^* \cdot W^\delta \tag{11}$$

We can improve Eq. (8).

Property 19 *Let $C \subseteq X^*$ be a prefix code and $W, V \subseteq C^*$. Then*

$$W \cdot V^\delta \subseteq (W \cdot V)^\delta \subseteq W \cdot V^\delta \cup (W^\delta \cap (C^* \cdot V)^\omega) .$$

Proof. If $\beta \in (W \cdot V)^\delta \setminus W \cdot V^\delta$ then $\beta \in W^\delta$, that is, there are infinitely many prefixes w_i of β in W. To each w_i belongs a $v_i \in V$ such that $w_i \cdot v_i$ is a prefix of β.

Choose the family $(w_i)_{i \in \mathbb{N}}$ in such a way that $|w_{j+1}| > |w_j \cdot v_j|$. Since C is a prefix code and $w_{j+1}, w_j, v_j \in C^*$ there is a $u_{j+1} \in C^*$ such that $w_j \cdot v_j \cdot u_{j+1} = w_{j+1}$. Hence, $\beta = w_1 \cdot v_1 \cdot u_2 \cdot v_2 \cdot \ldots \cdot u_j \cdot v_j \cdot \ldots \in (C^* \cdot V)^\omega$. ❑

Remark. In Property 19 it is important that C is indeed a prefix code. Consider e.g. the suffix code[5] $C := \{b, ba\}$. We obtain for $W := C^*$ and $V := \{b\}$ the proper inclusion $C^\omega = (C^* \cdot b)^\delta \supset C^\omega \cap (C^* \cdot b)^\omega$.

[4]The name δ-limit is due to the fact that an ω-language $F \subseteq X^\omega$ is a $\mathbf{G}_\delta$-set in CANTOR-space if and only if there is a language $W \subseteq X^*$ such that $F = W^\delta$.

[5]This code has, in addition, a delay of decipherability of 1, that is, whenever $w_1 \cdot w_2 \sqsubseteq w_1' \cdot w_2'$ for $w_1, w_2, w_1', w_2' \in C$ then $w_1 = w_1'$.

As a consequence of Property 19 we obtain that for a prefix code $C \subseteq X^*$ and $V \subseteq C^*$ it holds

$$(C^* \cdot V)^\delta = (C^* \cdot V)^\omega \cup C^* \cdot V^\delta \,. \tag{12}$$

We derive two further identities linking the operations of ω-power and δ-limit for languages of a special shape $C^* \cdot V$ or $W \cdot C^*$ where $C \subseteq X^*$ is a prefix code and $W, V \subseteq C^*$.

To this end let $\mathrm{Min}(W) := W \setminus W \cdot (X^* \setminus \{e\})$ be the set of minimal words with respect to "$\sqsubseteq$" in a language W.

$$(C^* \cdot V)^\omega \;=\; \big(\mathrm{Min}(C^* \cdot V)^*\big)^\delta \tag{13}$$

$$(W \cdot C^*)^\omega \;=\; \Big(W \cdot C^* \cdot \mathrm{Min}(W)\Big)^\delta \tag{14}$$

The proof can be easily transferred from the proof in the special case $C = X$ which can be found e.g. in [Pe85, Lt88].

In studying the relations between the ω-power and the δ-limit it is interesting to investigate as an intermediate operation the infinite intersection

$$\mathcal{D}(W) := \bigcap_{i \in \mathbb{N}} (W \setminus \{e\})^i \cdot X^\omega \,.$$

Though the assumption $W^\omega = \bigcap_{i \in \mathbb{N}} (W \setminus \{e\})^i \cdot X^\omega$ is tempting, it is well-known that in general W^ω and $\mathcal{D}(W)$ do not coincide. It holds only the obvious inclusion

$$W^\omega \subseteq \mathcal{D}(W) \subseteq (W^*)^\delta \,. \tag{15}$$

Next we give some examples which show that for both inclusions equality as well as proper inclusion in Eq. (15) may hold, independently of each other.

First we observe that Eq. (8) implies $W^\omega = \mathcal{D}(W) = (W^*)^\delta$ whenever $W^\delta \subseteq W^\omega$. Thus, in particular, the equality $W^\omega = \mathcal{D}(W) = (W^*)^\delta$ holds if W is finite or W is a prefix code (in these cases $W^\delta = \emptyset$).

In connection with the equality $W^\omega = \mathcal{D}(W) = (W^*)^\delta$ we mention the following connection to BOREL-classes.

Property 20 *If W^ω is closed then $W^\omega = \mathcal{D}(W) = (W^*)^\delta$, and if $W^\omega = \mathcal{D}(W)$ then W^ω is a $\mathbf{G}_\delta$-set.*

Proof. In virtue of Corollary 14 W^ω is closed iff $ls\, W \subseteq W^\omega$. Since $W^\delta \subseteq ls\, W$ the first assertion follows from Eq. (8). The second assertion follows from the definition of $\mathcal{D}(W)$. ❑

Our next example shows that proper inclusion in both cases is possible. Moreover it gives examples of regular languages of special form (one being a suffix code, the othe being prefix-closed) whose ω-power is not a $\mathbf{G}_\delta$-set.

Example 6 ([Pa81]) *Consider the suffix code $C_6 := \{a\} \cup c \cdot \{a,b\}^* \cdot b$.*
Then $cbaba^2ba^3 \ldots \in \mathcal{D}(C_6) \setminus C_6^\omega$ and $cb^\omega \in (C_6^*)^\delta \setminus \mathcal{D}(C_6)$, that is, $C_6^\omega \subset \mathcal{D}(C_6) \subset (C_6^*)^\delta$.

Moreover the intersection of C_6^ω with the closed set $c\cdot\{a,b\}^\omega \subseteq \{a,b,c\}^\omega$ satisfies $C_6^\omega \cap c\cdot\{a,b\}^\omega = c\cdot\{a,b\}^*\cdot b\cdot a^\omega \in \mathbf{F}_\sigma \setminus \mathbf{G}_\delta$. Hence $C_6^\omega \notin \mathbf{G}_\delta$.

We continue this example with the prefix-closure of C_6, $W_6 := \mathbf{A}(C_6) = \{e,a\}\cup c\cdot\{a,b\}^*$. Here we have similarly $W_6^\omega \cap c\cdot\{a,b\}^\omega = c\cdot\{a,b\}^*\cdot a^\omega \in \mathbf{F}_\sigma\setminus\mathbf{G}_\delta$, and $W_6^\omega \notin \mathbf{G}_\delta$.

Observe that as for C_6 it holds $cbaba^2ba^3\ldots \in \mathcal{D}(W_6)\setminus W_6^\omega$, and $cb^\omega \in (W_6^*)^\delta \setminus \mathcal{D}(W_6)$.

The purpose of the next example (due to WAGNER and WECHSUNG, cf. [St86, Example 3]) is twofold. First it shows that $\mathcal{D}(W) = (W^*)^\delta$ while $W^\omega \subset \mathcal{D}(W)$, and second the proper inclusion holds although W^ω is a $\mathbf{G}_\delta$-set.

Example 7 *Let $w_1 := a$ and $w_{i+1} := w_i^i\cdot b^i\cdot a$ for $i\geq 1$. Then $w_i \sqsubseteq w_i^i \sqsubset w_{i+1}$. Put $C_7 := \{w_i : i\geq 1\}$. It holds $C_7^\delta = \{\eta\}$ where $w_i^i \sqsubset \eta$ for all $i \geq 1$. Thus $\eta\in\mathcal{D}(C_7)$, but $\eta\notin C_7^\omega$. Moreover in [St86, Theorem 7 and Example 3] it is shown that C_7^ω is a $\mathbf{G}_\delta$-set.*

In view of the general identities $\mathcal{D}(W) = W^*\cdot\mathcal{D}(W)$ and $(W^*)^\delta = \mathcal{D}(W)\cup W^*\cdot W^\delta$, we have $C_7^*\cdot\eta \subseteq \mathcal{D}(C_7)$ whence the final conclusion $C_7^\omega \subset \mathcal{D}(C_7) = (C_7^*)^\delta$.

It should be noted that in view of Corollary 23 below the language in Example 7 cannot be chosen regular.

The fourth possibility can be verified again by regular languages.

Example 8 *Consider $C_8 := b\cdot a^*$ which is a (suffix) code having a delay of decipherability of 1.*

Hence, $C_8^\omega = \mathcal{D}(C_8)$ by Theorem 8 of [St86]. Since $b\cdot a^\omega \in C_8^\delta\setminus C_8^\omega$ we have $C_8^\omega = \mathcal{D}(C_8) \subset (C_8^*)^\delta$.

In Theorem 8 of [St86] it is shown that for codes $C\subseteq X^*$ having a bounded delay of decipherability[6] the identity $C^\omega = \mathcal{D}(C)$ holds. We present another class of languages for which this identity is true. Since $C_8^\omega = (b\cdot\{a,b\}^*)^\omega$ the subsequent lemma will also prove that $C_8^\omega = \mathcal{D}(C_8)$.

Lemma 21 *Let $W\subseteq X^*$ and $W = W\cdot X^*$. Then $W^\omega = \bigcap_{i\in\mathbb{N}} W^i\cdot X^\omega$.*

Proof. If $e\in W$ the assertion is clear.

Let $e\notin W$ and $\eta\in\bigcap_{i\in\mathbb{N}} W^i\cdot X^\omega$. We construct inductively a factorization $\eta = w_1\cdot v_1\cdots v_{i-1}\cdot w_i\cdot v_i\cdots$ where $w_i\in W$ and $v_i\in X^*$.

We start with an arbitrary $w_1\in W$ for which $w_1\sqsubset\eta$. Having defined $w_1\cdot v_1\cdots v_{i-1}\cdot w_i\sqsubset\eta$ let $\ell_i := |w_1\cdot v_1\cdots v_{i-1}\cdot w_i| + 1$. Since $\eta\in W^{\ell_i}\cdot X^\omega$, we have $w_1^{(\ell_i)}\cdots w_{\ell_i}^{(\ell_i)}\sqsubset\eta$ for words $w_1^{(\ell_i)},\ldots,w_{\ell_i}^{(\ell_i)}\in W$. By the choice of ℓ_i it follows $w_1\cdot v_1\cdots v_{i-1}\cdot w_i \sqsubseteq w_1^{(\ell_i)}\cdots w_{\ell_i-1}^{(\ell_i)}$. Define $v_i\in X^*$ such that $w_1\cdot v_1\cdots v_{i-1}\cdot w_i\cdot v_i = w_1^{(\ell_i)}\cdots w_{\ell_i-1}^{(\ell_i)}$ and $w_{i+1} := w_{\ell_i}^{(\ell_i)}$. ❑

A tight relation between W^ω and $\mathcal{D}(W)$ is given by the following lemma.

Lemma 22 *Let $v\cdot w^\omega\in\mathcal{D}(W)$ be an ultimately periodic sequence. Then $v\cdot w^\omega \in W^\omega$.*

[6] For codes having a bounded delay of decipherability see also [BP85] or [Sa81].

Proof. Let $v\cdot w^{\omega}\in\mathcal{D}(W)$. Then for every $i\in\mathbb{N}$ there is a prefix $u_1\cdots u_i$ of $v\cdot w^{\omega}$ such that $u_j\in W^*\setminus\{e\}$. Let u_1 be longer than v and let $i>|w|$. Then there are $j,k\leq i$ with $j<k$ such that $|u_1\cdots u_k|-|u_1\cdots u_j|$ is divisible by $|w|$. Hence $v\cdot w^{\omega}=u_1\cdots u_j\cdot(u_{j+1}\cdots u_k)^{\omega}$. ❑

Since regular ω-languages are characterized by their ultimately periodic ω-words, $\mathcal{D}(W)$ is not regular if W^{ω} is regular and $\mathcal{D}(W)\neq W^{\omega}$. Moreover, we have the following.

Corollary 23 *If W is regular and $\mathcal{D}(W)=(W^*)^{\delta}$ then $W^{\omega}=(W^*)^{\delta}$.*

Next we charcterize ω-power languages in several BOREL-classes. A first result for closed sets has been obtained in Corollary 14. We start with the BOREL-class $\mathbf{G}_{\delta}$. To this end we need the following operation. As in [St87b] we call

$$W\rhd V:=\{v:v\in V\wedge\exists W(w\in W\wedge w\sqsubseteq v\wedge\forall u(w\sqsubset u\sqsubset v\to u\notin V))\}$$

the *continuation* of the language W to the language V. In other words $W\rhd V$ consists of all those words in V which are minimal (w.r.t. "$\sqsubseteq$") prolongations of words in W. The following properties of the operation "$\rhd$" are shown in [St87b].

$$(W\rhd V)^{\delta}=W^{\delta}\cap V^{\delta}\tag{16}$$

Property 24 *$W\rhd V$ is a regular language if W and V are regular.*

Lemma 25 *An ω-power language W^{ω} is a $\mathbf{G}_{\delta}$-set if and only if there is a $V\subseteq W^*$ such that $W^{\omega}=(V^*)^{\delta}$. If, moreover, W is regular then V can be chosen to be also regular.*

Proof. The "if"-part is evident from the above remark on δ-limits.
If $W^{\omega}\subseteq X^{\omega}$ is a $\mathbf{G}_{\delta}$-set then there is a language $U\subseteq X^*$ such that $W^{\omega}=U^{\delta}$. Now set $V:=U\rhd W^*$. We obtain from Eq. (16) that $V^{\delta}=U^{\delta}\cap(W^*)^{\delta}=W^{\omega}$. Then in virtue of $V\subseteq W^*$ the assertion $(V^*)^{\delta}=V^{\omega}\cup V^*\cdot V^{\delta}=W^{\omega}$ follows.
The additional part on the regularity of V follows from Property 24 and the fact that U can be chosen also as a regular language provided W^{ω} is a regular ω-language. ❑

Now we turn to the ω-power languages which are open ω-languages.

Lemma 26 *An ω-power language $V^{\omega}\subseteq X^{\omega}$ is open if and only if there is a language $W\subseteq V^*$ such that $V^{\omega}=W^{\omega}=(W\cdot X^*)^{\omega}=W\cdot X^{\omega}$.*

Proof. Clearly, our condition is sufficient.
If $V^{\omega}\subseteq X^{\omega}$ is open there is a language $V'\subseteq X^*$ with $V^{\omega}=V'\cdot X^{\omega}$. Define

$$W:=V'\cdot X^*\cap(V^*\setminus\{e\})\,.$$

Obviously, $V^{\omega}=V'\cdot X^{\omega}\subseteq W\cdot X^{\omega}$. As the inclusions $W^{\omega}\subseteq(W\cdot X^*)^{\omega}\subseteq W\cdot X^{\omega}$ are evident, it remains to show that $V^{\omega}\subseteq W^{\omega}$.
Let $\xi=v_1\cdots v_i\cdots$ where $v_i\in V$, $v_i\neq e$. Because of $V^{\omega}=V'\cdot X^{\omega}$ it holds $\xi\in v'\cdot X^{\omega}$ for some $v'\in V'$. Then $v'\sqsubseteq v_1\cdots v_{|v'|}$ and, by construction, $v_1\cdots v_{|v'|}\in W$ and $v_{|v'|+1}\cdots v_i\cdots\in V^{\omega}$. Thus $V^{\omega}\subseteq W\cdot V^{\omega}$, and the assertion follows from Eq. (1). ❑

For ω-power languages of the form $(W\cdot X^*)^{\omega}$ we have the following necessary and sufficient conditions to be open.

Lemma 27 *Let $W \subseteq X^* \setminus \{e\}$ be a nonempty language. Then the following conditions are equivalent:*

1. *$(W \cdot X^*)^\omega$ contains a nonempty open subset.*
2. *$X^* \cdot W$ contains a finite maximal prefix code.*
3. *$(W \cdot X^*)^\omega = W \cdot X^\omega$.*

Proof. 3. $\Rightarrow$ 1. is obvious.
2. $\Rightarrow$ 3. Let $X^* \cdot W$ contain a finite maximal prefix code C. Then $X^\omega = C^\omega \subseteq (X^* \cdot W)^\omega$, whence $(W \cdot X^*)^\omega = W \cdot (X^* \cdot W)^\omega = W \cdot X^\omega$.
1. $\Rightarrow$ 2. First we observe that $\xi \in (W \cdot X^*)^\omega$ iff it has some $w_0 \in W$ as prefix and contains infinitely many nonoverlapping subwords $w_i \in W$, that is, ξ has the form $\xi = w_0 \cdot v_0 \cdot w_1 \cdot v_1 \cdots w_i \cdot v_i \cdots$ where $v_i \in X^*$.

Let now $u \cdot X^\omega \subseteq (W \cdot X^*)^\omega$ for some $u \in X^*$. Then every $\zeta \in X^\omega$ has the form $\zeta = v_0 \cdot w_1 \cdot v_1 \cdots w_i \cdot v_i \cdots$ where $w_i \in W$ and $v_i \in X^*$. Consequently, every $\zeta \in X^\omega$ has a prefix in $X^* \cdot W$. Thus $X^* \cdot W \cdot X^\omega = X^\omega$, which is equivalent to Condition 2. ❑

This lemma allows us to present examples of ω-power languages which are open but not closed and which are neither open nor closed but a union of an open and a closed set, respectively.

Property 28 *Let $\eta \in X^\omega \setminus \{x^\omega : x \in X\}$. Then $X^\omega \setminus \{\eta\}$ is an open nonclosed ω-power language.*

Proof. It is evident that $X^\omega \setminus \{\eta\}$ is an open nonclosed subset in CANTOR-space. Moreover, $X^\omega \setminus \{\eta\} = (X^* \setminus \mathbf{A}(\eta)) \cdot X^\omega$.

Since $\eta \notin \{x^\omega : x \in X\}$, it has a prefix $a^n \cdot b$ (say) where $a, b \in X$, $a \neq b$ and $n > 0$. Consequently, $(X \setminus \{a\}) \cup \{a^{n+1}\} \subseteq X^* \setminus \mathbf{A}(\eta)$. Thus X^{n+1} is a finite maximal prefix code contained in $X^* \cdot (X^* \setminus \mathbf{A}(\eta))$, and $X^\omega \setminus \{\eta\} = ((X^* \setminus \mathbf{A}(\eta)) \cdot X^*)^\omega$ follows from Lemma 27. ❑

We conclude with an example of an ω-power language which is neither open nor closed, but as a union of an open and a closed ω-language a set in a low level BOREL-class.

Example 9 *Let $C := \{a\} \cup \{bab\}^* \cdot bbb$. Then for $X := \{a, b\}$ the language $X^* \cdot C$ contains $\{a, ba, bba, bbb\}$ – a maximal prefix code. Hence $(C \cdot X^*)^\omega = C \cdot X^\omega$ is open and, since C is an infinite prefix code, $C \cdot X^\omega$ is not closed.*

Take the prefix code C and consider $F := (C \cdot X^* \cup \{baa\})^\omega$. Due to the identity $(V \cup W)^\omega = (W^* \cdot V)^\omega \cup (W^* \cdot V)^* \cdot W^\omega$ we obtain $F = (\{baa\}^* \cdot C \cdot X^*)^\omega \cup (\{baa\}^* \cdot C \cdot X^*)^* \cdot (baa)^\omega$.

Now we calculate $(\{baa\}^* \cdot C \cdot X^*)^\omega = \{baa\}^* \cdot (C \cdot X^* \cdot \{baa\}^*)^\omega = \{baa\}^* \cdot (C \cdot X^*)^\omega = \{baa\}^* \cdot C \cdot X^\omega$, and $(\{baa\}^* \cdot C \cdot X^*)^* = \{e\} \cup \{baa\}^* \cdot C \cdot X^*$.

Thus $F = \{baa\}^* \cdot C \cdot X^\omega \cup (\{baa\}^* \cdot C \cdot X^*) \cdot (baa)^\omega \cup \{(baa)^\omega\} = \{baa\}^* \cdot C \cdot X^\omega \cup \{(baa)^\omega\}$ is a union of the open set $\{baa\}^* \cdot C \cdot X^\omega$ with the closed set $\{(baa)^\omega\}$. It remains to show that F is neither open nor closed.

To this end observe that $(baa)^\omega \notin \{baa\}^* \cdot C \cdot X^\omega$, thus F is not open, and that $(bab)^\omega \in \mathcal{C}(F) \setminus F$.

5 Topological density

In this section we study the density of regular and finite-state ω-languages in ω-power languages. It turns out that in this case density and subwords are closely related.

Topological density is based on the following notion. A set F is *nowhere dense* in $E \subseteq X^\omega$ provided $\mathcal{C}(E \setminus \mathcal{C}(F)) = \mathcal{C}(E)$, that is, if $\mathcal{C}(F)$ does not contain a nonempty subset of the form $E \cap w \cdot X^\omega$. This condition can be formulated as follows.

Lemma 29 *A set $F \subseteq X^\omega$ is nowhere dense in E iff for every $v \in \mathbf{A}(E)$ there is a $w \in X^*$ such that $v \cdot w \in \mathbf{A}(E)$ and $v \cdot w \cdot X^\omega \cap F = \emptyset$.*

Cast in the language of prefixes, our Lemma 29 asserts, that F is not nowhere dense in $E \neq \emptyset$ if and only if there is a $w \in \mathbf{A}(E)$ such that $E/w \subseteq \mathcal{C}(F)/w$. From the following equation

$$\mathcal{C}(E \setminus \mathcal{C}(F)) = \mathcal{C}(E \setminus (\mathcal{C}(F) \cap E)) = \mathcal{C}(\mathcal{C}(E) \setminus \mathcal{C}(F)) \tag{17}$$

we see that F is nowhere dense in E iff F is nowhere dense in $\mathcal{C}(E)$ and iff $(\mathcal{C}(F) \cap E)$ is nowhere dense in E.

A subset $F \subseteq X^\omega$ is called nowhere dense if it is nowhere dense in X^ω. For finite-state nowhere dense ω-languages we have the following.

Lemma 30 ([St76,80b]) *A finite-state set $F \in X^\omega$ is nowhere dense iff there is a pattern $w \in X^*$ such that $F \subseteq X^\omega \setminus X^* \cdot w \cdot X^\omega$.*

The aim of this section is to generalize the result of Lemma 30 to finite-state ω-languages nowhere dense in an ω-power language W^ω.

We obtain the following version of Lemma 29.

Corollary 31 *Let $W \subseteq X^*$. Then $F \subseteq X^\omega$ is nowhere dense in W^ω if and only if for every $v \in W^*$ there is a $w \in W^*$ such that $v \cdot w \cdot X^\omega \cap F = \emptyset$.*

Cast again in the language of prefixes, we have that F is not nowhere dense in an ω-power language W^ω if and only if there is a $w \in W^*$ such that $W^\omega/w \subseteq \mathcal{C}(F)/w$. We obtain the following necessary and sufficient conditions for a finite-state ω-language to be nowhere dense in an ω-power language.

Lemma 32 *Let $W \subseteq X^*$, and let $F \subseteq X^\omega$ be a finite-state ω-language. Then the following conditions are equivalent.*

1. *F is nowhere dense in W^ω.*
2. $\forall u(u \in W^* \Rightarrow F/u$ *is nowhere dense in* $W^\omega)$
3. $\forall w(w \in Stab(\mathcal{C}(W^\omega)) \cup \{e\} \Rightarrow F/w$ *is nowhere dense in* $W^\omega)$
4. $\forall v(v \in X^* \Rightarrow (\mathcal{C}(F) \cap W^\omega)/v$ *is nowhere dense in* $W^\omega)$

5. $\forall v(v \in X^* \Rightarrow (\mathcal{C}(F) \cap \mathcal{C}(W^\omega))/v$ *is nowhere dense in* $W^\omega)$

Remark. Observe that, in general, the stabilizer of $\mathcal{C}(W^\omega)$, $Stab(\mathcal{C}(W^\omega))$ contains $Stab(W^\omega) \supseteq W^* \setminus \{e\}$. Thus Condition 3 shows more states F/w of F to be nowhere dense in W^ω than Condition 2.

Proof. The implications 5. $\Rightarrow$ 4., 4. $\Rightarrow$ 1., 3. $\Rightarrow$ 2., and 2. $\Rightarrow$ 1. are obvious.

To conclude the proof, it suffices to show 1. $\Rightarrow$ 3. and 3. $\Rightarrow$ 5.. To this end assume first that Condition 5 does not hold, that is, there is a $v \in X^*$ such that $(\mathcal{C}(F) \cap \mathcal{C}(W^\omega))/v$ is not nowhere dense in W^ω. Then according to Corollary 31 there is a $w \in W^*$ satisfying $(\mathcal{C}(F) \cap \mathcal{C}(W^\omega))/v \cdot w \supseteq \mathcal{C}(W^\omega)/w$. Since $w \in W^* \subseteq Stab(\mathcal{C}(W^\omega)) \cup \{e\}$, we have $\mathcal{C}(W^\omega)/w \supseteq \mathcal{C}(W^\omega)$. Consequently, $u := v \cdot w \in Stab(\mathcal{C}(W^\omega)) \cup \{e\}$, and $\mathcal{C}(F)/u \supseteq W^\omega$ which shows that F/u is not nowhere dense in W^ω.

Now assume Condition 3 to be violated, that is, let F/w be not nowhere dense in W^ω for some $w \in Stab(\mathcal{C}(W^\omega)) \cup \{e\}$. According to Corollary 31 there is a $v \in W^* \subseteq Stab(\mathcal{C}(W^\omega)) \cup \{e\}$ such that $\mathcal{C}(F)/w \cdot v \supseteq \mathcal{C}(W^\omega)/v$. Consequently, $u := w \cdot v \in Stab(\mathcal{C}(W^\omega))$.

Since F is finite-state, there are $n, k \geq 1$ such that $F/u^n = F/u^{n+k}$. Hence $\mathcal{C}(W^\omega) \subseteq \mathcal{C}(F)/u$ implies $\mathcal{C}(W^\omega)/u^{n+k-1} \subseteq \mathcal{C}(F)/u^{n+k} = \mathcal{C}(F)/u^n$.

Now observe that $\mathcal{C}(W^\omega)/u^n \subseteq \mathcal{C}(W^\omega)/u^{n+k-1} \subseteq \mathcal{C}(F)/u^n$, because $u \in Stab(\mathcal{C}(W^\omega)) \cup \{e\}$, what proves our assertion. ❑

As a consequence of Lemma 32 we show the announced generalization of Lemma 30 that for finite-state ω-languages nowhere dense in ω-power languages W^ω there are patterns, that is subwords appearing in the ω-power language W^ω which do not appear in the finite-state ω-language F. Those patterns can be shown to belong to W^*. Due to the possiblity that $F \not\subseteq \mathcal{C}(W^\omega)$ we have to distinguish two cases.

Theorem 33 *Let* $F \subseteq X^\omega$ *be finite-state, and let* $W^* \subseteq X^*$.

1. F *is nowhere dense in* W^ω *iff there is a* $w \in W^*$ *such that* $\mathcal{C}(F) \cap \mathcal{C}(W^\omega) \subseteq \mathcal{C}(W^\omega) \setminus W^* \cdot w \cdot X^\omega$.

2. *If* $F \subseteq \mathcal{C}(W^\omega)$ *then* F *is nowhere dense in* W^ω *iff there is a* $u \in W^*$ *such that* $F \subseteq \mathcal{C}(W^\omega) \setminus X^* \cdot u \cdot X^\omega$.

Proof. 1. If F is finite-state and nowhere dense in W^ω then according to Lemma 32.2 the set $F' := \bigcup_{u \in W^*} F/u$ as a finite union of sets nowhere dense in W^ω is again nowhere dense in W^ω. Hence, there is a $w \in W^*$ such that $F' \cap w \cdot X^\omega = \emptyset$.
Assume that $F \cap W^* \cdot w \cdot X^\omega \neq \emptyset$. Then there is some $v \in W^*$ such that $F \cap v \cdot w \cdot X^\omega = v \cdot (F/v) \cap v \cdot w \cdot X^\omega \neq \emptyset$, which contradicts the fact that $F' \supseteq F/v$ and $w \cdot X^\omega$ are disjoint.

To prove the converse direction, suppose F to be not nowhere dense in W^ω, that is, according to Lemma 32.2 and Corollary 31 there is some $u \in W^*$ such that $\mathcal{C}(F)/u \cdot w \supseteq \mathcal{C}(W^\omega)/w \supseteq \mathcal{C}(W^\omega)$ for some $w \in W^*$. Hence, $\mathbf{A}(F) \supseteq w \cdot u \cdot W^*$ and there is no $v \in W^*$ with $F \cap u \cdot w \cdot v \cdot X^\omega = \emptyset$.

2. In view of Lemma 32.5 from $\mathcal{C}(F) \subseteq \mathcal{C}(W^\omega)$ we obtain that the finite union $F'' := \bigcup_{u \in X^*} F/u$ is also nowhere dense in W^ω provided F is nowhere dense in W^ω.

Now the proof proceeds as in 1. The converse direction of the second part is an immediate consequence of the first part. ❏

For ω-powers of codes we obtain the following corollary to Theorem 33.1.

Corollary 34 *Let $F \subseteq X^\omega$ be finite-state, and let $C \subseteq X^*$ be a code. If F is nowhere dense in C^ω then there are a $k > 0$ and a word $u \in C^k$ such that $F \cap \mathcal{C}(C^\omega) \subseteq \mathcal{C}\big((C^k \setminus \{u\})^\omega\big)$*

The converse statement is, however, not true in general. Consider e.g. the suffix code $C := \bigcup_{n\in\mathbb{N}}\{a,b\}^n \cdot b \cdot a^n$. Here $\mathcal{C}((C^k \setminus \{u\})^\omega) = \mathcal{C}(C^\omega) = \{a,b\}^\omega$ for every pair $k > 0$ and $u \in C^k$, but $F := \{a,b\}^\omega$ is dense in C^ω.

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