Momente eines rotierten Quaders

Berechnung der Momente eines beliebig rotierten Quaders, gegeben durch den x-Achsenbereich $[v_u, v_o]$ und den y-Achsenbereich $[u_u, u_o]$, den er in einem nicht rotierten Koordinatensystem einnimmt, und den Winkel α, um den das Koordinatensystem gegenüber dem nicht-rotierten Koordinatensystem gegen den Uhrzeigersinn gedreht ist.

$$m_{pq}(\alpha, v_u, v_o, u_u, u_o) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^p quader(x,y) dxdy$$
$$\text{mit } quader(x,y) := f, \text{ wenn (x,y) innerhalb des gegebenen Quaders liegt}$$

Koordinatentransformation mit $\begin{pmatrix} u \\ v \end{pmatrix} = R(-\alpha)\begin{pmatrix} x \\ y \end{pmatrix}$ und $R(\alpha) := \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}$:

$$\Leftrightarrow m_{pq}(\alpha, v_u, v_o, u_u, u_o) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p(u,v) y^q(u,v) quader(x(u,v), y(u,v))|D| dudv$$

$$\text{wobei } D = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos(-\alpha) & \sin(-\alpha) \\ -\sin(-\alpha) & \cos(-\alpha) \end{vmatrix} = \cos^2(-\alpha) + \sin^2(-\alpha) = 1$$

$$\begin{aligned}\Leftrightarrow m_{pq}(\alpha, v_u, v_o, u_u, u_o) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (u\cos(-\alpha) + v\sin(-\alpha))^p (v\cos(-\alpha) - u\sin(-\alpha))^q \\ &\qquad quader(x(u,v), y(u,v)) dudv \\ &= f\int_{u_u}^{u_o}\int_{v_u}^{v_o} (u\cos(-\alpha) + v\sin(-\alpha))^p (v\cos(-\alpha) - u\sin(-\alpha))^q dudv\end{aligned}$$

Also:

$$\begin{aligned}
m_{00}(\alpha, v_u, v_o, u_u, u_o) =& f(v_o - v_u)(u_o - u_u) \\
m_{10}(\alpha, v_u, v_o, u_u, u_o) =& \frac{f}{2}(u_o - u_u)(v_o - v_u)((u_o + u_u)\cos(\alpha) - (v_o + v_u)\sin(\alpha)) \\
m_{01}(\alpha, v_u, v_o, u_u, u_o) =& \frac{f}{2}(u_o - u_u)(v_o - v_u)((v_o + v_u)\cos(\alpha) + (u_o + u_u)\sin(\alpha)) \\
m_{11}(\alpha, v_u, v_o, u_u, u_o) =& \frac{f}{12}(u_o - u_u)(v_o - v_u)\big(3(u_o + u_u)(v_o + v_u)\cos(2\alpha) \\
&+ 2\left({u_o}^2 + u_o u_u + {u_u}^2 - {v_o}^2 - v_o v_u - {v_u}^2\right)\sin(2\alpha)\big)
\end{aligned}$$

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\begin{aligned}
m_{20}(\alpha, v_u, v_o, u_u, u_o) =& \frac{f}{12}\left(u_o-u_u\right)\left(v_o-v_u\right)\left(2\left(u_o^2+u_o u_u+u_u^2+{v_o}^2+v_o v_u+{v_u}^2\right)\right.\\
&+\ 2\left({u_o}^2+u_o u_u+{u_u}^2-{v_o}^2-v_o v_u-{v_u}^2\right)\cos(2\alpha)\\
&\left.-\ 3\left(u_o+u_u\right)\left(v_o+v_u\right)\sin(2\alpha)\right)
\end{aligned}
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\begin{aligned}
m_{02}(\alpha, v_u, v_o, u_u, u_o) =& \frac{f}{12}\left(u_o-u_u\right)\left(v_o-v_u\right)\Big(4\left({v_o}^2+v_o v_u+\ {v_u}^2\right)\cos(\alpha)^2\\
&+\ 4\left({u_o}^2+u_o u_u+{u_u}^2\right)\sin(\alpha)^2+3\left(u_o+u_u\right)\left(v_o+v_u\right)\sin(2\alpha)\Big)
\end{aligned}
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\begin{aligned}
m_{21}(\alpha, v_u, v_o, u_u, u_o) =& \frac{f}{24}\left(\left(u_o-\ u_u\right)\left({v_o}^2-{v_u}^2\right)\cos(\alpha)\left({u_o}^2+u_o u_u+{u_u}^2+3{v_o}^2+3{v_u}^2\right.\right.\\
&\left.-\ 3\left({v_o}^2+{v_u}^2\right)\cos(2\alpha)\right)+3\left({u_o}^3-{u_u}^3\right)\left({v_o}^2-{v_u}^2\right)\cos(3\alpha)\\
&+\ 6\left({u_o}^4-{u_u}^4\right)\left(v_o-v_u\right)\cos(\alpha)^2\sin(\alpha)\\
&\left.+\ \left({u_o}^2-{u_u}^2\right)\left({v_o}^3-\ {v_u}^3\right)\left(\sin(\alpha)-3\sin(3\alpha)\right)\right)
\end{aligned}
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